A Property of a Classical Solution of the Nonlinear Mass Transport Equation $u_t = u_{xx}/1 + u_x^2$

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A mechanism of smoothing due to evaporation-condensation of the roughly perturbed surface of solid is described by Mullins, W. W. in terms of the Cauchy problem (P) in R^1 (real line) of a nonlinear parabolic equation for $u(x, t)$ representing the evolution of the profile of the surface.

$$u_t = u_{xx}/1 + u_x^2, \quad (x, t) \in R^1 \times (0, \infty); \quad u(x, 0) = \alpha(x), \quad x \in R^1 \quad (\text{P})$$

In the present paper, it is demonstrated that each peak height of the initial surface $\alpha(x)$ in Mullins' model (P) does not increase with time.

I. Introduction

Mullins, W. W.¹⁾ reported that a mechanism of surface smoothing in solid due to evaporation-condensation could be formulated as the Cauchy problem (P) in R^1 (real line) of a nonlinear mass transport equation describing the evolution of the profile $u(x, t)$ of the surface of solid.

$$\left. \begin{aligned} u_t &= u_{xx}/1 + u_x^2, & (x, t) &\in R^1 \times (0, \infty) \\ u(x, 0) &= \alpha(x) \quad (\alpha \in \mathcal{B}^1(R^1)^*), & x &\in R^1 \end{aligned} \right\} \quad (\text{P})$$

Mullins verified the physical validities of the model (P), i.e. the smoothing properties of solutions of (P), by making the problem (P) analytically solvable in the classical sense²⁾, through the linearization of the nonlinear equation in (P) by small slope approximation (ssa) characterized by the relation $|u_x(x, t)| \ll 1$ for any point $(x, t) \in R^1 \times (0, \infty)$. But, necessarily, this linearization based on ssa has restricted the extent of application of this model. In fact, the unique solution of (P) under ssa is described as

$$u(x, t) = \int_{R^1} H(x-y, t) \alpha(y) dy, \quad H(p, q) = \frac{1}{2\sqrt{\pi q}} \exp \frac{-p^2}{4q}.$$

The operation, permitted by Lebesgue's convergence theorem,

$$\lim_{t \downarrow 0} u_x(x, t) = \lim_{t \downarrow 0} \frac{1}{\sqrt{\pi}} \int_{R^1} e^{-k^2} \cdot \alpha'(x - 2\sqrt{t}k) dk = \alpha'(x) \quad \begin{array}{l} \text{compact} \\ \text{convergence} \end{array}$$

demands that the slope of the peak in the initial surface also should be small enough, namely, $|\alpha'(x)| \ll 1$.

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* $\mathcal{B}^1(R^1) = \{f; f \in C^1(R^1), |f(x), f'(x)| < \infty\}$.

In the present paper, without employing this linearization, by demonstrating that each peak height of the initial surface $\alpha(x)$ does not increase with time, we shall indirectly show that the nonlinear model (P) does not contradict the smoothing properties of the surface of solid.

II. A Property of a Solution of a Cauchy Problem for $u_t = u_{xx}/1 + u_x^2$

Consider a Cauchy problem (P*) for a more general nonlinear parabolic equation.

$$\left. \begin{aligned} u_t &= F(u_x, u_{xx}) \quad (F \in C^1(R^{2+1})), \quad (x, t) \in R^1 \times (0, \infty) \\ u(x, 0) &= \alpha(x), \quad x \in R^1 \end{aligned} \right\} \quad (\text{P}^*)$$

Let x_0 be a point at which $\alpha(x)$ attains the strict relative maximum (local maximum), i.e. there exist a peak whose top is located at x_0 ; and let the curve C in $R^1 \times [0, t_f]$ be a part of a trajectory drawn by the migration with time of the peak top initially located at x_0 (Fig. 1). Suppose C is characterized by a differentiable function $g(t)$ on the interval $[0, t_f]$ as indicated in (1).

$$C = \{(x, t); x = g(t) \quad (x_0 = g(0)), t \in [0, t_f]\} \quad (1)$$

From the definition of the set C , at any point in C , the following relations hold.

$$u_x(x, t) = 0, \quad u_{xx}(x, t) < 0 \quad (2)$$

[THEOREM] *Let the conditions*

$$\frac{\partial F(p, q)}{\partial q} > 0, \quad F(0, 0) = 0 \quad (3)$$

hold for the right hand side $F(p, q)$ of the nonlinear equation in (P*). Suppose, for the classical solution²⁾ $u(x, t)$ of (P*), there exists a set C characterized by (1) and (2). Then,

$$u(x, t) \leq \alpha(x_0), \quad (x, t) \in C \setminus \{(x_0, 0)\}^\# \quad (4)$$

Figure 1

[†] R^2 denotes the real plane.

[#] $A \setminus B = \{x; x \in A, x \notin B\}$

Before demonstrating this theorem, we prepare a lemma for a linear parabolic equation.

[LEMMA] Let $u(x, t)$ be a classical solution of the following linear problem (LP).

$$\left. \begin{aligned} u_t - a(x, t)u_{xx} + b(x, t)u_x &= f(x, t) \quad (a(x, t) > 0, |f(x, t)| < \infty), \\ (x, t) &\in R^1 \times (0, \infty) \end{aligned} \right\} \quad (\text{LP})$$

$$u(x, 0) = \alpha(x), \quad x \in R^1$$

Suppose, for $u(x, t)$, there exists a set C with a form of (1) and, on the set C , $u(x, t)$ satisfies the relations (2). Then the estimate (5) is valid for any point $(x^*, t^*) \in C \setminus \{(x_0, 0)\}$.

$$u(x^*, t^*) \leq \max \left\{ \sup_{C \setminus \{(x_0, 0)\}} \frac{f(x, t)e^{\lambda(t^*-t)}}{\lambda}, \quad \alpha(x_0)e^{\lambda t^*} \right\} \quad (5)$$

where λ is an arbitrary positive number.

[PROOF OF LEMMA] Set $v(x, t) = u(x, t)e^{-\lambda t}$. On the set $C \setminus \{(x_0, 0)\}$, $v(x, t)$ satisfies the linear equation

$$v_t - a(x, t)v_{xx} + b(x, t)v_x + \lambda v = f(x, t)e^{-\lambda t}.$$

Since $v(x, t)$ is continuous on the compact set C in R^2 , $v(x, t)$ attains its maximum on C at some point (ξ, η) in C . The point (ξ, η) lies either in $C \setminus \{(x_0, 0)\}$ or in the singleton set $\{(x_0, 0)\}$.

First, let the point (ξ, η) exist in the set $C \setminus \{(x_0, 0)\}$. From the relations (1) and (2), we obtain

$$v_x(\xi, \eta)g'(\eta) + v_t(\xi, \eta) \geq 0.$$

Since $v_x(\xi, \eta) = u_x(\xi, \eta)e^{-\lambda\eta} = 0$, we get

$$v_t(\xi, \eta) \geq 0.$$

Namely, at the point (ξ, η) , it must hold that

$$v_x(\xi, \eta) = 0, \quad v_{xx}(\xi, \eta) < 0, \quad v_t(\xi, \eta) \geq 0.$$

Therefore, for any $(\tilde{x}, \tilde{t}) \in C \setminus \{(x_0, 0)\}$,

$$v(\tilde{x}, \tilde{t}) \leq v(\xi, \eta) \leq \sup_{C \setminus \{(x_0, 0)\}} \frac{f(x, t)e^{-\lambda t}}{\lambda}.$$

Taking (x^*, t^*) as $(\tilde{x}, \tilde{t})$ and replacing $v(x^*, t^*)$ by $u(x^*, t^*)$, we can obtain an estimate

$$u(x^*, t^*) \leq \sup_{C \setminus \{(x_0, 0)\}} \frac{f(x, t)e^{\lambda(t^*-t)}}{\lambda}.$$

If the maximum on C is attained at the point $(x_0, 0)$, the estimate

$$u(x^*, t^*) \leq \alpha(x_0)e^{\lambda t^*}$$

is valid.

These two estimates are unified to (5). $\square$

[PROOF OF THEOREM]

Let $u(x, t)$ and $v(x, t)$ be a classical solution of the problem (P*) and (P**) respectively.

$$\left. \begin{aligned} v_t &= F(v_x, v_{xx}), & (x, t) &\in R^1 \times (0, \infty) \\ v(x, 0) &= \alpha^*(x), & v &\in R^1 \end{aligned} \right\} \quad (P^{**})$$

Taking the differences between two nonlinear equations, we can get the following linear homogeneous equation for $w(x, t) = u(x, t) - v(x, t)$.

$$\begin{aligned} & u_t - F(u_x, u_{xx}) - v_t + F(v_x, v_{xx}) \\ &= w_t - \int_0^1 \frac{dF(hu_x + (1-h)v_x, hu_{xx} + (1-h)v_{xx})}{dh} dh \\ &= w_t - \left\{ w_{xx} \int_0^1 \frac{\partial F(hu_x + (1-h)v_x, hu_{xx} + (1-h)v_{xx})}{\partial q} dh \right. \\ & \quad \left. + w_x \int_0^1 \frac{\partial F(hu_x + (1-h)v_x, hu_{xx} + (1-h)v_{xx})}{\partial p} dh \right\} = 0. \end{aligned}$$

If we set $\alpha^* \equiv 0$, it is easy to see from the second relation in (3) that $v(x, t) \equiv 0$ is a solution of (P**), that is, $w(x, t)$ identically equals $u(x, t)$. Then, $u(x, t)$, the solution of (P*) must satisfy a homogeneous linear equation

$$u_t - u_{xx} \int_0^1 \frac{\partial F(hu_x, hu_{xx})}{\partial q} dh - u_x \int_0^1 \frac{\partial F(hu_x, hu_{xx})}{\partial p} dh = 0.$$

Applying the estimate (5) in Lemma to this function $u(x, t)$, we can easily obtain, for any point $(x^*, t^*) \in C \setminus \{(x_0, 0)\}$,

$$u(x^*, t^*) \leq \max\{0, \alpha(x_0)e^{\lambda t^*}\}.$$

Since we may take $\alpha(x_0)$ positive* and λ is arbitrary, it holds that

$$u(x^*, t^*) \leq \alpha(x_0).$$

Thus, we can obtain the estimate (4). $\square$

Since $F \in C^1(R^2)$, $\partial F / \partial q = 1/(1+p^2) > 0$ and $F(0, 0) = 0$, the conditions required in (P*) and (3) are well satisfied for the right hand side of the nonlinear mass transport equation $u_t = F(u_x, u_{xx}) = u_{xx}/(1+u_x^2)$. Therefore, if the solution $u(x, t)$ of the Cauchy problem (P) has such a set as C characterized by the relations (1) and (2), the desired result

$$u(x, t) \leq \alpha(x_0), \quad (x, t) \in C \setminus \{(x_0, 0)\}$$

* Even if $\alpha(x_0) \leq 0$, the same results are obtained. See Appendix.

is obtained.

Appendix

If $u(x, t)$ is a solution of the problem (P*), $\tilde{u}(x, t) = u(x, t) + C$ (C is an arbitrary constant) is a solution of the following Cauchy problem ($\tilde{P}^*$).

$$\left. \begin{aligned} \tilde{u}_t &= F(\tilde{u}_x, \tilde{u}_{xx}), & (x, t) &\in R^1 \times (0, \infty) \\ \tilde{u}(x, 0) &= \alpha(x) + C, & x &\in R^1 \end{aligned} \right\} \quad (\tilde{P}^*)$$

In the same manner as in the text, we can obtain the estimate

$$u(x^*, t^*) + C \leq \tilde{u}(x^*, t^*) \leq \max\{0, (\alpha(x_0) + C)e^{\lambda t^*}\}$$

where C is a positive constant such that $|\alpha(x_0)| < C$. Hence,

$$u(x^*, t^*) \leq \alpha(x_0)e^{\lambda t^*} + C(e^{\lambda t^*} - 1).$$

Since λ is arbitrary, desired relation

$$u(x^*, t^*) \leq \alpha(x_0)$$

holds.

References

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